Quantum Mechanics I

Week 8 (Solutions)

Spring Semester 2025

1 Expectation Values for the Harmonic Oscillator

Consider the one-dimensional simple harmonic oscillator.

For the calculations in this exercise, we wish to avoid any integrals. Thus, we will express the position and momentum operators in terms of the ladder operators as we defined them in the lecture:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right). \tag{1.1}$$

The position and momentum operators are:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right), \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a}). \tag{1.2}$$

We will also use a property of the ladder operators when acting on the number states, namely

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{1.3}$$

as well as the orthonormality of the number states $\langle n|m\rangle = \delta_{n,m}$. Using these relations, we can deduce the action of the position and momentum operators on the number states:

$$\hat{x}\,|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\Big[\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\Big], \tag{1.4}$$

$$\hat{p}\,|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\Big[\sqrt{n+1}\,|n+1\rangle - \sqrt{n}\,|n-1\rangle\Big]. \tag{1.5}$$

A. Evaluate the following matrix elements:

(a) $\langle m|x|n\rangle$

$$\begin{aligned}\langle m|x|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\,\langle m|(a+a^\dagger)|n\rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}\right].\end{aligned}$$

(b) $\langle m|p|n\rangle$

$$\langle m|p|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\,\langle m|(a^\dagger - a)|n\rangle$$
$$= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right].$$

(c) $\langle m|\{x,p\}|n\rangle$

$$\langle m|\{x,p\}|n\rangle = \langle m|xp|n\rangle + \langle m|px|n\rangle$$
$$= i\sqrt{\frac{\hbar\, m\,\omega}{2}}\left[\sqrt{n+1}\,\langle m|x|n+1\rangle - \sqrt{n}\,\langle m|x|n-1\rangle\right] +$$
$$+\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\langle m|p|n-1\rangle + \sqrt{n+1}\,\langle m|p|n+1\rangle\right]$$
$$= i\frac{\hbar}{2}\left[(n+1)\delta_{m,n} + \sqrt{(n+1)(n+2)}\delta_{m,n+2} - \sqrt{n(n-1)}\delta_{m,n-2} - n\delta_{m,n}\right] +$$
$$+ i\frac{\hbar}{2}\left[n\delta_{m,n} - \sqrt{n(n-1)}\delta_{m,n-2} + \sqrt{(n+1)(n+2)}\delta_{m,n+2} - (n+1)\delta_{m,n}\right]$$
$$= i\hbar\left[\sqrt{(n+1)(n+2)}\delta_{m,n+2} - \sqrt{n(n-1)}\delta_{m,n-2}\right].$$

(d) $\langle m|x^2|n\rangle$

$$\langle m|x^2|n\rangle = \sqrt{\frac{\hbar}{2\,m\,\omega}}\left[\sqrt{n}\,\langle m|x|n-1\rangle + \sqrt{n+1}\,\langle m|x|n+1\rangle\right]$$
$$= \frac{\hbar}{2\,m\,\omega}\left[\sqrt{n(n-1)}\,\delta_{n-2,m} + (2n+1)\,\delta_{n,m} + \sqrt{(n+1)(n+2)}\,\delta_{n+2,m}\right].$$

(e) $\langle m|p^2|n\rangle$

$$\langle m|p^2|n\rangle = i\sqrt{\frac{\hbar\, m\,\omega}{2}}\left[\sqrt{n+1}\,\langle m|p|n+1\rangle - \sqrt{n}\,\langle m|p|n-1\rangle\right]$$
$$= -\frac{\hbar\, m\,\omega}{2}\left[\sqrt{(n+1)(n+2)}\,\delta_{n+2,m} - (2n+1)\,\delta_{n,m} + \sqrt{n(n-1)}\,\delta_{n-2,m}\right].$$

where $|n\rangle\,, |m\rangle$ represent number states of the harmonic oscillator. Use the ladder operators to avoid any integrals.

B. From classical physics, the virial theorem states that:

$$\left\langle \frac{p^2}{m}\right\rangle = \left\langle x\frac{dV}{dx}\right\rangle. \tag{1.6}$$

Check that the virial theorem holds for the expectation values of the kinetic and the potential energy taken with respect to an energy eigenstate.

For the harmonic oscillator potential:

$$V(x) = \frac{1}{2}m\omega^2 x^2 \quad \Rightarrow \quad \frac{dV}{dx} = m\omega^2 x\,.$$

Using our result from Question A(d) for the state $|n\rangle$, we find

$$\left\langle \frac{p^2}{m} \right\rangle = \frac{\hbar\omega}{2}\,(2n+1) = \hbar\omega\Big(n+\tfrac{1}{2}\Big), \quad \text{and} \quad \left\langle x\,\frac{dV}{dx} \right\rangle = \frac{\hbar\omega}{2}\,(2n+1) = \hbar\omega\Big(n+\tfrac{1}{2}\Big).$$

and the virial theorem is indeed satisfied.

2 Linear combination of eigenstates of the harmonic oscillator

Consider a linear harmonic oscillator. Let ψ_0 and ψ_1 be the respective real eigenfunctions of the ground state and the first excited state. Let

$$\psi = A\psi_0 + B\psi_1$$

be the wave function of the oscillator at a given time, where A and B are real values, and properly normalized.

(a) Calculate the expectation value of the position operator as a function of ψ_0 and ψ_1.

The expectation value of $\hat{x}$ is given by

$$\begin{aligned}\langle \hat{x} \rangle &= \int \psi^*(x)x\psi(x)dx = \int x(A\psi_0 + B\psi_1)^2 dx \\ &= A^2\langle\psi_0|\hat{x}|\psi_0\rangle + B^2\langle\psi_1|\hat{x}|\psi_1\rangle + 2AB\langle\psi_0|\hat{x}|\psi_1\rangle \\ &= 2AB\langle\psi_0|\hat{x}|\psi_1\rangle. \end{aligned} \tag{2.1}$$

where $\langle\psi_0|\hat{x}|\psi_0\rangle = 0$ and $\langle\psi_1|\hat{x}|\psi_1\rangle = 0$ for the harmonic oscillator.

(b) Find the coefficients A and B which maximize or minimize $\langle \hat{x} \rangle$.

Using the orthonormality condition of the eigenfunctions of the harmonic oscillator, we find

$$\int |\psi(x)|^2 dx = \int (A\psi_0(x) + B\psi_1(x))^2 dx = 1 \quad \Rightarrow \quad A^2 + B^2 = 1. \tag{2.2}$$

Since $\langle\psi_0|\psi_1\rangle = 0$, we can rewrite our result from the previous question as

$$\begin{aligned}\langle \hat{x} \rangle &= 2AB\langle\psi_0|\hat{x}|\psi_1\rangle = (1 - 1 + 2AB)\langle\psi_0|\hat{x}|\psi_1\rangle \\ &= [1 - (A-B)^2]\langle\psi_0|\hat{x}|\psi_1\rangle. \end{aligned} \tag{2.3}$$

To find the values of A and B for which $\langle \hat{x} \rangle$ is extremal, we consider the function:

$$f = AB = A\sqrt{1 - A^2}, \tag{2.4}$$

and differentiate:

$$\frac{df}{dA} = \frac{1 - 2A^2}{\sqrt{1 - A^2}} = 0 \quad \Rightarrow \quad A = \pm\frac{1}{\sqrt{2}}. \tag{2.5}$$

Therefore, using the result of Eq. (2.3), we find that for $A = B = \frac{1}{\sqrt{2}}$, $\langle \hat{x} \rangle$ is maximized, and for $A = -B = -\frac{1}{\sqrt{2}}$, $\langle \hat{x} \rangle$ is minimized.

Another option is to use the method of Lagrange multipliers. This implies that we need to minimize the function $A^2 + B^2 - \lambda AB$ where λ is a Lagrange multiplier. The equation has to be solved together with the constraint $A^2 + B^2 = 1$. Differentiating with respect to A and B gives $2A - \lambda B = 0$, $2B - \lambda A = 0$.

The solutions are $\langle \hat{x} \rangle$ stationary are $B = \pm A$ and, due to the normalization condition, give as before, $A = B = \frac{1}{\sqrt{2}}$ and $A = -B = -\frac{1}{\sqrt{2}}$.

3 Time evolution of the harmonic oscillator

We consider a quantum harmonic oscillator in one dimension which is characterized by a mass m and angular frequency ω.

(a) We prepare the system in the state

$$|\Psi_1(t = 0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle),$$

where $|n\rangle$ $(n = 0, 1, \ldots)$ describes the n-th excited eigenstate of the oscillator $(a^\dagger a|n\rangle = n|n\rangle)$. Give the expression of $|\Psi_1(t)\rangle$ corresponding to a state of the system at some time t.

The time-evolution of $|\Psi_1(t = 0)\rangle$ is governed by the time-dependent Schrödinger equation:

$$i\hbar\frac{d}{dt}|\Psi_1(t)\rangle = \hat{H}|\Psi_1(t)\rangle. \tag{3.1}$$

Since the Hamiltonian is time-independent, the solution to the above equation is given by

$$\begin{aligned} |\Psi_1(t)\rangle &= e^{-\frac{i}{\hbar}\hat{H}t}|\Psi_1(0)\rangle \\ &= \frac{1}{\sqrt{2}}\left[e^{-i\omega t/2}|0\rangle + e^{-i7\omega t/2}|3\rangle\right] \\ &= \frac{1}{\sqrt{2}}e^{-i\omega t/2}\left[|0\rangle + e^{-i3\omega t}|3\rangle\right]. \end{aligned} \tag{3.2}$$

(b) Deduce that the evolution of the system is periodic. Give the period of the state of the system, i.e., the minimum time after which the expectation value of any observable will be the same as that at $t = 0$.

We deduce that the temporal evolution of the system is periodic. The global phase $e^{-i\omega t/2}$ which multiplies the terms between brackets has no influence on the physics of the evolution of the system. The system returns to the original state when the phase in front of $|3\rangle$ is one, i.e.

$$e^{i3\omega t} = 1 \quad \Rightarrow \quad 3\omega t = 2n\pi, \tag{3.3}$$

where n is an integer number. The period then is trivially obtained:

$$T_1 = \frac{2\pi}{3\omega}. \tag{3.4}$$

(c) Consider now an initial state

$$|\Psi_2(t=0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + e^{i\alpha}|4\rangle).$$

Redo questions (a) and (b) for the state $|\Psi_2(t=0)\rangle$.

In a similar fashion as in question (a), the time-evolution of $|\Psi_2(0)\rangle$ is given by:

$$\begin{aligned}|\Psi_2(t)\rangle &= e^{-\frac{i}{\hbar}\hat{H}t}|\Psi_2(0)\rangle \\ &= \frac{1}{\sqrt{2}}e^{-i3\omega t/2}\Big[|1\rangle + e^{i\alpha}e^{-i3\omega t}|4\rangle\Big].\end{aligned} \tag{3.5}$$

Using very similar arguments as in the previous question, we find:

$$T_2 = \frac{2\pi}{3\omega}. \tag{3.6}$$

(d) Consider finally an initial state of the form of a linear combination of the two previous states,

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}\big[|\Psi_1(t=0)\rangle + |\Psi_2(t=0)\rangle\big].$$

What is the period of the state of the system?

At a time t, the state $|\Psi(t)\rangle$ is given by:

$$\begin{aligned}|\Psi(t)\rangle &= e^{-\frac{i}{\hbar}\hat{H}t}|\Psi(0)\rangle \\ &= \frac{1}{\sqrt{2}}\Big[e^{-i\omega t/2}|0\rangle + e^{-i7\omega t/2}|3\rangle + e^{-i3\omega t/2}|1\rangle + e^{i\alpha}e^{-i9\omega t/2}|4\rangle\Big] \\ &= \frac{1}{\sqrt{2}}e^{-i\omega t/2}\Big[|0\rangle + e^{-i3\omega t}|3\rangle + e^{-i\omega t}|1\rangle + e^{i\alpha}e^{-i4\omega t}|4\rangle\Big]\end{aligned} \tag{3.7}$$

Of the four terms in square brackets, three terms contain complex exponentials with different periods. Specifically,

$$T_{|1\rangle} \equiv T' = \frac{2\pi}{\omega}, \quad T_{|3\rangle} = \frac{2\pi}{3\omega} = \frac{T'}{3}, \quad T_{|4\rangle} = \frac{2\pi}{4\omega} = \frac{T'}{4}. \tag{3.8}$$

Given this observation, the state ket $|\Psi\rangle$ returns back to its original form after time

$$T = \frac{2\pi}{\omega}, \tag{3.9}$$

which corresponds to the period we were looking for.

It is interesting to remark that the linear combination of the two states $|\Psi_1(t)\rangle$ and $|\Psi_2(t)\rangle$, which both have the same period individually, does not necessarily give a state which evolves with that same period. In fact, the period of the total state is three times that of the system described by the individual states $|\Psi_1(t)\rangle$ and $|\Psi_2(t)\rangle$.

(e) Consider now an *arbitrary* initial state. Show that the time-evolution of the state is always periodic under $t \to t + T$ with $T = 2\pi/\omega$ (the classical period of oscillation of the harmonic oscillator). Note that this does not exclude a higher periodicity under $t \to t + nT$ with $n \geq 1$ an integer.

Any initial state can be expanded as

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \ , \tag{3.10}$$

with $\sum_{n=0}^{\infty} |c_n|^2 = 1$. The time-evolved state at time t in the Schrödinger picture is

$$|\psi(t)\rangle = e^{-i\omega t/2} \sum_{n=0}^{\infty} c_n e^{-in\omega t} |n\rangle \ . \tag{3.11}$$

If $t \to t + T$ with $T = 2\pi/\omega$, the wavefunction becomes

$$|\psi(t+T)\rangle = e^{-i\omega t/2} e^{-i\pi} \sum_{n=0}^{\infty} c_n e^{-in\omega t} |n\rangle = -|\psi(t)\rangle \ . \tag{3.12}$$

So we see that any wavefunction after a time T returns equal to itself, up to a phase factor (in this case, a simple minus sign). The minus sign does not change the probabilities and the averages of operators calculated at time $t + T$. So the system is periodic with period T.

4 Oscillator in an Electric Field

The Hamiltonian describing a certain system can be approximated by an oscillator of mass m and frequency ω. The bound particle has electric charge q. Initially, the system is in its ground state. The system is placed in an external electric field $\mathcal{E}$.

(a) Write down the new Hamiltonian.

The new Hamiltonian is

$$H = H_0 - d\,\mathcal{E} \;=\; \frac{p^2}{2m} \;+\; \frac{1}{2}\,m\omega^2\,x^2 \;-\; \mathcal{E}\,q\,x,$$

which differs from H_0 only by a shift in the coordinate (minus an energy constant):

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \mathcal{E}qx = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\Big(x - \frac{\mathcal{E}\,q}{m\omega^2}\Big)^2 - \frac{1}{2}\frac{\mathcal{E}^2 q^2}{m\omega^2}.$$

(b) Determine its energy spectrum.

To determine the energies, we define a new shifted position operator

$$\tilde{x} \equiv x - \frac{\mathcal{E}\,q}{m\omega^2}, \tag{4.1}$$

thus the Hamiltonian becomes:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\tilde{x}^2 - \frac{1}{2}\frac{\mathcal{E}^2 q^2}{m\omega^2}. \tag{4.2}$$

Then, similarly as in the zero-field case, we define ladder operators $a, a^\dagger$ in terms of the momentum and the shifted position operator. The Hamiltonian is now expressed in terms of the ladder operators as:

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) - \frac{1}{2}\frac{\mathcal{E}^2\,q^2}{m\,\omega^2}. \tag{4.3}$$

Thus, the eigenvalues of H are:

$$E_n \;=\; \hbar\omega\big(n + \tfrac{1}{2}\big) \;-\; \frac{1}{2}\frac{\mathcal{E}^2\,q^2}{m\,\omega^2}.$$

(c) Find the position representation of the new ground state of the system. Express your result in terms of the quantities

$$\ell \;=\; \sqrt{\frac{\hbar}{m\,\omega}}, \qquad b \;=\; \frac{\mathcal{E}\,q}{m\,\omega^2}.$$

Recall the wavefunction for the old ground state:

$$\psi_0(x) = \pi^{-\frac{1}{4}}\,\frac{1}{\sqrt{\ell}}\,\exp\!\Big(-\tfrac{1}{2}\,\tfrac{x^2}{\ell^2}\Big),$$

where we have introduced the length ℓ. Now, the new ground state is simply the shifted by b compared to the old ground state,

$$\psi_{\tilde{0}}(x) = \pi^{-\frac{1}{4}}\,\frac{1}{\sqrt{\ell}}\,\exp\!\Big(-\tfrac{1}{2}\,\tfrac{(x-b)^2}{\ell^2}\Big).$$

This can be shown by simply solving the equation $\langle x| \, a \, |\tilde{0}\rangle = 0$ using the new definition of the lowering operator in terms of the shifted position operator.

The excited states are obtained by applying the raising operator on the ground state.

(d) What is the probability that, once the electric field is switched on, the system is found in the (new) ground state?

The probability that the system is found in the new ground state $\psi_{\tilde{0}}(x)$ is:

$$P = \left|\langle 0 \mid \tilde{0}\rangle\right|^2 = \left|\int_{-\infty}^{+\infty} \psi_0(x)\, \psi_{\tilde{0}}(x)\, dx\right|^2 = \exp\Big(-\frac{b^2}{2\,\ell^2}\Big).$$

In the second equality, we have used the resolution of identity in the position basis.

In the case of zero electric field, we have $b = 0$, and thus the probability is unity, as expected.

(e) If the system is in the new ground state, what is the value of its average dipole moment?

The dipole operator is $d = q\,x$. By shifting the variable to $z = x - b$, the new Hamiltonian is identical to the old one (up to a constant). Therefore,

$$\langle \tilde{0} \mid q\,x \mid \tilde{0}\rangle = \langle \tilde{0} \mid q\,z \mid \tilde{0}\rangle \;+\; \langle \tilde{0} \mid q\,b \mid \tilde{0}\rangle = q\,b \;=\; \frac{q^2}{m\,\omega^2}\,\mathcal{E}.$$

Therefore, applying an electric field to our quantum system induces a non-zero dipole moment, which is consistent with our expectations from classical electrodynamics.

5 Harmonic oscillator in the Heisenberg and the Schrödinger pictures

(a) Solve the Heisenberg equations of motion for the time-dependent operators $\hat{x}(t)$, $\hat{p}(t)$ in the case of a single harmonic oscillator.

The Heisenberg equations of motion read

$$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}[\hat{H}, \hat{x}(t)] = \frac{i}{\hbar}\left[\frac{\hat{p}^2(t)}{2m}, \hat{x}(t)\right] \tag{5.1}$$

$$\frac{d}{dt}\hat{p}(t) = \frac{i}{\hbar}[\hat{H}, \hat{p}(t)] = \frac{i}{\hbar}\left[\frac{m\omega^2\hat{x}^2(t)}{2}, \hat{p}(t)\right] \;. \tag{5.2}$$

The commutators at time t can be calculated using that $[\hat{x}(t), \hat{p}(t)] = [\hat{x}(0), \hat{p}(0)] = i\hbar$. This gives:

$$\frac{d}{dt}\hat{x}(t) = \frac{\hat{p}(t)}{m} \;, \qquad \frac{d}{dt}\hat{p}(t) = -m\omega^2\hat{x}(t) \;. \tag{5.3}$$

The equations are identical to those of a classical harmonic oscillator. (The same property is valid for any Hamiltonian of the form $H = \hat{p}^2/2m + V(\hat{x})$). The solution of the equations of motion is

$$\begin{aligned} \hat{x}(t) &= \cos(\omega t)\hat{x}(0) + \sin(\omega t)\hat{p}(0)/(m\omega) \\ \hat{p}(t) &= -\sin(\omega t)m\omega\hat{x}(0) + \cos(\omega t)\hat{p}(0) \ . \end{aligned} \tag{5.4}$$

The operators at time $t = 0$ coincide with the corresponding operators in the Schrödinger representation.

(b) Calculate the average values $\langle x \rangle$, and $\langle p \rangle$, for an arbitrary initial state and at an arbitrary time t using the Heisenberg representation.

In the Heisenberg picture, we can calculate the time-dependent averages using the corresponding time-dependent operators:

$$\begin{aligned} \langle \hat{x}(t) \rangle &= \cos(\omega t)\langle \psi_0|\hat{x}(0)|\psi_0\rangle + \sin(\omega t)\frac{\langle \psi_0|\hat{p}(0)|\psi_0\rangle}{m\omega} \ , \\ \langle \hat{p}(t) \rangle &= -m\omega\sin(\omega t)\langle \psi_0|\hat{x}(0)|\psi_0\rangle + \cos(\omega t)\langle \psi_0|p(0)|\psi_0\rangle \ , \end{aligned} \tag{5.5}$$

Here, $|\psi_0\rangle$ is the state vector in the Heisenberg representation (which is time-independent).

(c) Repeat the calculation in the Schrödinger representation using the basis of stationary states and show that equivalent results are obtained.

To recover the same results in the Schrödinger representation we can decompose the state vector at $t = 0$ as

$$|\psi_0\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \ . \tag{5.6}$$

The time-evolved state is then

$$|\psi(t)\rangle = e^{-i\omega t/2}\sum_{n=0}^{\infty} c_n e^{-in\omega t}|n\rangle \ . \tag{5.7}$$

The averages $\langle \hat{x}(t) \rangle$, $\langle \hat{p}(t) \rangle$ can then be calculated as

$$\begin{aligned} \langle \hat{x}(t) \rangle &= \langle \psi(t)|\hat{x}|\psi(t)\rangle = \sum_{n,k} e^{i(k-n)\omega t}c_k^* c_n \langle k|\hat{x}|n\rangle \ , \\ \langle \hat{p}(t) \rangle &= \langle \psi(t)|\hat{p}|\psi(t)\rangle = \sum_{n,k} e^{i(k-n)\omega t}c_k^* c_n \langle k|\hat{p}|n\rangle \ , \end{aligned} \tag{5.8}$$

The matrix elements $\langle k|\hat{x}|n\rangle$, $\langle k|\hat{p}|n\rangle$ are nonzero only if $n = k \pm 1$. Thus $\cos((k-n)\omega t) = \cos(\omega t)$ and $\sin((k-n)\omega t) = (k-n)\sin\omega t$.

Then we can rewrite

$$\begin{aligned} \langle \hat{x}(t) \rangle &= \sum_{n,k} c_k^* c_n [\cos(\omega t)\langle k|\hat{x}|n\rangle + i(k-n)\sin(\omega t)\langle k|\hat{x}|n\rangle] \\ \langle \hat{p}(t) \rangle &= \sum_{n,k} c_k^* c_n [\cos(\omega t)\langle k|\hat{p}|n\rangle + i(k-n)\sin(\omega t)\langle m|\hat{p}|n\rangle] \ . \end{aligned} \tag{5.9}$$

We can now two another properties of the matrix elements, which can be shown using the results of exercise 1: $\langle k|\hat{p}|n\rangle = i(k-n)m\omega\langle k|\hat{x}|n\rangle$, $\langle k|\hat{x}|n\rangle = -i(k-n)\langle k|\hat{p}|n\rangle/(m\omega)$. Substituting in the expression and using that for all nonzero matrix elements $(k-n)^2 = 1$ we find:

$$\begin{aligned}
\langle\hat{x}(t)\rangle &= \sum_{n,k} c_k^* c_n \left[\cos(\omega t)\langle k|\hat{x}|n\rangle + \frac{1}{m\omega}\sin(\omega t)\langle k|\hat{p}|n\rangle\right] \\
&= \langle\psi_0\left|\hat{x}\cos(\omega t) + (\hat{p}/(m\omega))\right|\psi_0\rangle \ , \\
\langle\hat{p}(t)\rangle &= \sum_{n,k} c_k^* c_n \left[-m\omega\sin(\omega t)\langle k|\hat{x}|n\rangle + \cos(\omega t)\langle k|\hat{p}|n\rangle\right] \\
& \langle\psi_0|(-m\omega\sin(\omega t)\hat{x} + \cos(\omega t)\hat{p})|\psi_0\rangle \ .
\end{aligned} \tag{5.10}$$

Identifying $\hat{x} = \hat{x}(0)$ and $\hat{p} = \hat{p}(0)$ we recover the equivalence with the result found in the Heisenberg picture.

In this calculation we checked explicitly and in a particular case the equivalence of the two calculations. The equivalence however, is fully general, and can be proved more generally and more quickly by identifying $\hat{x}(t) = \hat{U}^{-1}(t)\hat{x}\hat{U}$. In the Schrödinger representation the average is calculated as $\langle\psi(t)|\hat{x}|\psi(t)\rangle$ which inserting $\hat{U}(t)|\psi_0\rangle$ and $\langle\psi(t)| = \langle\psi_0|\hat{U}^\dagger(t) = \langle\psi_0|\hat{U}^{-1}(t)$ becomes $\langle\psi_0|\hat{U}^{-1}(t)\hat{x}\hat{U}(t)|\psi_0\rangle$. In the Heisenberg picture we assign the evolution to the operator: $\hat{x}(t) = \hat{U}^{-1}(t)\hat{x}\hat{U}(t)$ and we find the same result for the time-dependent average: $\langle\hat{x}(t)\rangle = \langle\psi_0|\hat{U}^{-1}(t)\hat{x}\hat{U}|\psi_0\rangle$.